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LETTER TO THE EDITOR

Laplace transform of a class of *G* functions

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Abstract. The Laplace transform of Meijer's G function with an argument proportional to an integer power of the integration variable is obtained in closed form. In particular this allows a straightforward evaluation of the Laplace transform of the Airy functions $\mathrm{Ai}(\pm x)$ and $\mathrm{Bi}(-x)$.

The Laplace transform of the Airy functions has occurred in a variety of physical problems, most recently in a study of surface state tunnelling in one-dimensional solids (Davison and Kolar 1985). These transforms have been obtained previously by several authors. Smith (1973) and Davison and Glasser (1982) used a successive integration of the Airy differential equation and the former author also mentions the applicability of the Fourier-Bessel series. On the other hand, as shown by Leach (1983), these results can also be derived from the integral representation of the Airy functions. The resulting transforms can be written as a linear combination of confluent hypergeometric functions.

In this letter it is shown that these transforms are special cases of the Laplace transform of Meijer's G function with argument proportional to x^k, $k \in \mathbb{N}$. The latter have not been found in any of the standard tables of Laplace transforms, except for the cases $k=1$ or 2. Since Meijer's G function includes many of the special functions of mathematics as particular cases (for an extensive list see Luke 1969) the results presented here have a very general applicability.

Meijer's G function is most conveniently defined as a Mellin-Barnes integral (Luke 1969):

$$G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right) = G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)$$
$$= (2\pi\mathrm{i})^{-1}\int_L \prod_{j=1}^{m}\Gamma(b_j-s))\prod_{j=1}^{n}\Gamma(1-a_j+s)$$
$$\times\left(\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)\right)^{-1} z^s\,\mathrm{d}s. \tag{1}$$

In this expression an empty product is to be interpreted as unity, $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$, and a_h and b_h are such that no pole in the first product of the expression coincides with a pole in the second product. The path of integration is to be chosen so that the former poles lie to the right of it and the latter ones lie to its left. Many interesting properties of this function are known and more details can be found in Luke's (1969) book. In the following we will merely need the following expansion in terms of generalised hypergeometric functions:

$$G^{m,n}_{p,q}\left(z\left|\begin{matrix}a_p\\b_q\end{matrix}\right.\right)=\sum_{h=1}^{m}\prod_{j=1}^{m}\Gamma(b_j-b_h)^*\prod_{j=1}^{n}\Gamma(1+b_h-a_j)$$
$$\times\left(\prod_{j=m+1}^{q}\Gamma(1+b_h-b_j)\prod_{j=n+1}^{p}\Gamma(a_j-b_h)\right)^{-1}z^{b_h}$$
$$\times{}_pF_{q-1}\left(\begin{matrix}1+b_h-a_p\\1+b_h-b_q^*\end{matrix}\left|(-1)^{p-m-n}z\right.\right)\tag{2}$$

valid when $p<q$ or $p=q$ and $|z|<1$, and no two of the b_h terms, $h=1,\ldots,n$, differ by an integer or zero. The asterisks in (2) denote that $j\neq h$, $q\neq h$ respectively.

The main result of this letter is the following identity:

$$\int_0^\infty\exp(-\beta x)x^{-\alpha}G^{m,n}_{p,q}\left(zx^k\left|\begin{matrix}a_p\\b_q\end{matrix}\right.\right)\mathrm{d}x$$
$$=\beta^{\alpha-1}(2\pi)^{(1-k)/2}k^{-\alpha+1/2}$$
$$\times G^{m,n+k}_{p+k,q}\left(\frac{zk^k}{\beta^k}\left|\begin{matrix}\alpha/k,\ldots,\dfrac{\alpha+k-1}{k},a_p\\b_q\end{matrix}\right.\right)\tag{3}$$

which is valid under the following conditions. Define

$$\delta=m+n-\tfrac{1}{2}(p+q)\tag{4}$$

then the following condition must hold:

$$\delta>0\qquad|\arg z|<\delta\pi\tag{5a}$$

or

$$\delta=0\qquad\arg z=0\tag{5b}$$

and

$$\mathrm{Re}\,(\beta)>0\tag{6}$$

and

$$\mathrm{Re}\,(b_j)>\frac{\alpha-1}{k}\qquad j=1,2,\ldots,m.\tag{7}$$

These conditions follow from the requirement that the integrand in (3) has proper behaviour near the origin and infinity in order to ensure convergence of the integral. In certain specific cases these conditions may be relaxed along the same lines as in Luke (1969, 5.6) but for the sake of conciseness we will not elaborate on this point. The result (3) can be proved by inserting the definition of the G function in the left-hand side and interchanging the order of the integrations, which is valid since the two integrals are absolutely convergent. The infinite integral is a gamma function which can be written by means of the multiplication formula and the remaining loop integral can be identified as a G function.

In order to obtain from this result the Laplace transform of the Airy functions we need the following identities which can easily be deduced from the list of special G functions in Luke (1969):

$$\mathrm{Ai}(x)=(2\pi)^{-1}3^{-1/6}G^{2,0}_{0,2}(x^3/9|\tfrac{1}{3},0)\tag{8}$$

$$\mathrm{Ai}(x)=3^{-2/3}[G_{0,2}^{1,0}(-x^3/9|0,\tfrac{1}{3})+G_{0,2}^{1,0}(-x^3/9|\tfrac{1}{3},0)] \tag{9}$$

and

$$\mathrm{Bi}(x)=3^{-1/6}[G_{0,2}^{1,0}(-x^3/9|0,\tfrac{1}{3})-G_{0,2}^{1,0}(-x^3/9|\tfrac{1}{3},0)]. \tag{10}$$

The first expression gives

$$\int_0^\infty \exp(-px)\mathrm{Ai}(x)\,\mathrm{d}x=(4\pi^2p)^{-1}3^{1/3}G_{3,2}^{2,3}\left(3p^{-3}\middle|\begin{matrix}0,\frac{1}{3},\frac{2}{3}\\ \frac{1}{3},0\end{matrix}\right). \tag{11}$$

The resulting G function can be expanded by means of (2) and after some further simplifications and using

$$\phi(a,b)=b^{1-a}(1-a)^{-1}{}_1F_1\left(\begin{matrix}1-a\\2-a\end{matrix}\middle|b\right) \tag{12}$$

we obtain

$$\int_0^\infty \exp(-px)\;\mathrm{Ai}(x)\,\mathrm{d}x=\tfrac{1}{3}\exp(-p^3/3)$$
$$\times[1+\phi(\tfrac{2}{3},p^3/3)/\Gamma(\tfrac{2}{3})-\phi(\tfrac{1}{3},p^3/3)/\Gamma(\tfrac{1}{3})] \tag{13}$$

in full agreement with Leach (1983). There is of course no Laplace transform for $\mathrm{Bi}(x)$ since the integrand becomes singular. In order to obtain the Laplace transform of $\mathrm{Ai}(-x)$ and $\mathrm{Bi}(-x)$ we have to start from (9) and (10) to find results identical to Leach (1983):

$$\int_0^\infty \exp(-px)\mathrm{Ai}(-x)\,\mathrm{d}x=\tfrac{1}{3}\exp(p^3/3)$$
$$\times[2-\gamma(\tfrac{1}{3},p^3/3)/\Gamma(\tfrac{1}{3})-\gamma(\tfrac{2}{3},p^3/3)/\Gamma(\tfrac{2}{3})] \tag{14}$$

$$\int_0^\infty \exp(-px)\;\mathrm{Bi}(-x)\,\mathrm{d}x=3^{-1/2}\exp(p^3/3)$$
$$\times[\gamma(\tfrac{1}{3},p^3/3)/\Gamma(\tfrac{1}{3})-\gamma(\tfrac{2}{3},p^3/3)/\Gamma(\tfrac{2}{3})] \tag{15}$$

where

$$\gamma(a,b)=a^{-1}b^a\,\mathrm{e}^{-b}{}_1F_1\left(\begin{matrix}1\\a+1\end{matrix}\middle|b\right) \tag{16}$$

is the incomplete gamma function. It is worth noting that we had to use (8) and not (9) in order to obtain (13) and conversely (9) and not (8) to obtain (14), the reason being the condition of validity (5*a*).

References

Davison S G and Glasser M L 1982 *J. Phys. A: Math. Gen.* **15** L463-5
Davison S G and Kolar M 1985 *J. Phys. C: Solid State Phys.* **18** 4581-90
Leach P G L 1983 *J. Phys. A: Math. Gen.* **16** L451-3
Luke Y L 1969 *The Special Functions and Their Approximations* vol 1 (New York: Academic)
Smith P 1973 *SIAM Rev.* **15** 796-804